Some Results on Prime Graphs

Shreemathi Adiga

M.Sc, M Phil.
Assistant Professor of Mathematics

Govt First Grade College, Koteshwara, Kundapura TQ
Udupi District, Karnataka, INDIA

Email id: adigashreemathi@gmail.com

Abstract:

This paper consists of some results on Prime graphs. We introduce the concept of Prime graphs and prove that Binary trees of different levels are Prime graphs. We extend the result to n-ary trees. Also we give some examples of nonprime graphs. We give an alternate proof for Pn and Cn which are proved as Prime graphs.

Now in this paper we prove that Binary trees of level 1, 2, 3, 4, are Prime graphs.

Trinary (3-ary) trees of level 1, 2, 3 are Prime graphs

Definition 1: Graph Labelling:

If the vertices of a graph are assigned values subject to certain conditions then it is known as Graph Labeling.

By a labeling of the vertices of the graph $G=(V,E)$, we mean a mapping $f: V \rightarrow A$ where $A$ is called the label Set; $A=\{1, 2, 3, \ldots, V(G)\}$ where $|V(G)|$ is the cardinality of the vertex Set. The Labelling of vertices is an injective mapping if different vertices have different labels.

An Injective Labeling is bijective if there are as many labels in $A$ as number of vertices.

Definition 2: Prime Graphs

A prime labeling of a graph $G$ is an injective function $f: V \rightarrow A$ where $A=\{1, 2, 3, \ldots, V(G)\}$ such that, for every pair of adjacent vertices $V_1$ and $V_2$, $\gcd(f(V_1), f(V_2)) = 1$.

The graph which admits a prime labelling is called a Prime graph.

Definition 3: Co prime Labeling:

Co prime labeling of a simple graph of order $n$ is a labeling in which adjacent vertices are given relatively prime. If the labels used are 1st $n$ prime numbers then this graph will be a Prime graph.
Remarks

- The concept of prime labeling was first introduced by Entringer.
- Tout et al.[4] discussed about Prime graphs and later many researchers studied about Prime graphs.
- Fee and Huang(5) have proved that Pn and K1n are Prime graphs.
- Lee et al(6) have proved that Wn is a Prime graphs iff n is even.
- Deretsky et al(7) have proved that Cn is a Prime graph.
- Vaidya and Kanani proved that the graph obtained by fusing (identifying) any two vertices U and V with d(u,v)≥ 3 of Cn (n ≥ 4) is a Prime graphs.
- Samir K.Vaidya, Udayan M and Prajapati have proved that the graph obtained by identifying any two vertices of K1n and Pn is a Prime graphs.
  Also they proved that if p is a prime and G is a Prime graph of order p then the graph obtained by identifying two vertices with label 1 and p is also a Prime graph.

Strongly Prime graphs:

A graph G is said to be a strongly Prime graph if for any vertex V of G, there exists a prime labeling f satisfying f(V)=1

- It is proved that Every path(Pn), cycle (Cn), K1n are strongly Prime graphs , Wn is strongly Prime graphs for every even positive integer n ≥ 4
- we prove that Binary trees of level 1,2,3,4 are Prime graphs.

Also we try to prove that trinary (3 ary) trees of level 1,2,3 are Prime graphs.

We try to extend the result for Banana trees, Caterpillars, Kmn , Divisor graphs and all trees.

Also we give Examples for a non tree which is a Prime graphs.

[Fu, Huang proved that trees with n ≤ 15 are prime graphs Pikhurdext[4] extended it for n ≤ 34]

Definition : Tree

A tree is a connected graph without any circuits.

Binary tree:

A binary tree is a tree in which there is exactly one vertex of degree 2 and all remaining vertices are of degree 1 or 3.

The vertex of degree ‘2’ is distinct from all others vertices, this vertex is regarded as a root. So binary trees are special cases of rooted trees.
**Definition:**

In a binary tree, a vertex \( v \), is said to be at level \( l \), if \( v_i \) is at a distance ‘\( l \)’ from the root.

The root is at level 0.

A binary tree having 17 vertices with 4 level is shown below.

There must be only one vertex at level 0,

Atmost 2 vertices at level 1

At most 8 vertices at level 3 etc

So maximum number of vertices possible in

1) 1 level binary tree is \( = 1 + 2^0 + 2^1 = 3 = 2^2 - 1 \)
2) 2 level binary tree is \( = 1 + 2 + 4 = 2^0 + 2^1 + 2^2 = 7 = 2^3 - 1 \)
3) 3 level binary tree is \( = 1 + 2^2 + 2^3 = 15 = 2^4 - 1 \)
4) 4 level binary tree is \( = 1 + 2^2 + 2^3 + 2^4 = 31 = 2^5 - 1 \)
5) In K level binary tree is \( = 1 + 2^2 + \ldots + 2^k = 2^{k+1} - 1 \)

**Height of a tree**

The maximum level \( l_{\text{max}} \) of any vertex in a binary tree is called the height of a tree.
Lmax=5                                      Height of above tree=5

Definition : Complete Binary tree

A Binary tree in which every vertex except the pendant vertices have 2 branches (2 childrens). i.e, Every level except possibly the the last is completely filled.

Definition : N-ary trees

A n-ary tree is a rooted tree in which each vertex has no more than ‘n’ children.

If K=2 it is a binary tree
If K=3, it is trinary or 3 a-ry tree.

Trinary trees

A tree in which only one vertex is of degree 3 and all the vertices adjacent to it have 3 branches.(Childrens) (Eg : fig 1 above)

A complete n-ary tree:

A n-ary tree is a tree in which each vertex has exactly n children.
Theorem

A complete Binary tree of level 1,2,3,4 are **Prime graphs**.

Proof

**level 1:**

A complete Binary tree of level 1 has 3 vertices

Define \( f: V(G) \rightarrow \{1,2,3\} \) as \( f(V_i)=i \) for \( i=1,2,3 \)

Eg:

Then \( f \) is an injection and \( \gcd(f(u), f(v))=1 \) for all pair of adjacent vertices.

**Level 2:**

It has \( 1+2+4=7 \) vertices : \( n=7 \)

\( \emptyset(7)=6 \)

All the numbers 1,2,3-----6 are relatively prime to 7.

The prime numbers among them are 2,3,5.

Define \( f: V(G) \rightarrow \{1,2,3-----7\} \) by

\( f(V_i)=1 \) if \( V_i \) is the root.

\( f(V_i) \) = an odd prime no. if \( V_i \) is in 1st level, \( 1<p \leq 7 \);

\( f(V_i) = \frac{q}{q}=n \) or \( q \) is even if \( V_i \) in the 2\(^{nd}\) level.
Then f is an injection and gcd(f(u), f(v)) = 1 for all pair of adjacent vertices.

**Level 3:**

A complete Binary tree with level 3 has $1+2+4+8 = 15$ vertices.

So $n = 15$. Let $V(G) = \{V_1, V_2, ..., V_{15}\}$

Prime no.s less than 15 are 2, 3, 5, 7, 11, 13.

Define: $f : V(G) \rightarrow \{1, 2, 3, \ldots, 15\}$ by:

- $f(V_i) = 1$ if $V_i$ is the root
- $f(V_i) = x$ if $V_i$ is in the 1st level where $2 \leq x < 15$ and x is even
- $f(V_i) = y$ if $V_i$ is in the 2nd level, where $y$ is an odd prime, $y \geq 5$
- $f(V_i) = Z$ if $V_i$ is in the 3rd level, where $3 \leq Z \leq 15$
  - $Z$ is not an odd prime, $Z \neq 1, Z \neq 3$

Then f is an injection and gcd(f(u), f(v)) = 1 for all pair of adjacent vertices.

**Level 4:**

A complete Binary tree with level 4 has $1+2+4+8+16 = 31$ vertices.

So $n = 31$, $V(G) = \{V_1, V_2, ..., V_{31}\}$

Let $V_1$ be the root.

Define $f : V(G) \rightarrow \{1, 2, 3, \ldots, 31\}$ by:

- $f(V_1) = 1$
- Define $f(V_i) = x$, Where $x$ is not a prime, $2 < x < 31$, if $V_i$ is in the 1st level
- Define $f(V_i) = y$, $y$ is any odd prime, if $V_i$ is in the 2nd level
- Define $f(V_i) = Z$, where $2 \leq Z \leq 31$, $Z$ is a prime except for one $V_i$, if $V_i$ is in 3rd level
- Define $f(V_i) = r$ where $2 < r < 31$, $r$ is a composite no, if $V_i$ is in 4th level

Then $f$ is an injection and gcd(f(u), f(v)) = 1 for all pair of adjacent vertices.

Hence the complete binary graphs of level 1, 2, 3, 4 are Prime graphs.
Now we give examples of 3-ary trees of level 1,2, which are prime graphs.
3-ary trees level 1

3-ary trees level 2

References
